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FINITE AMPLITUDE CONVECTION IN A TWO-COMPONENT FLUID SATURATED POROUS LAYER

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Abstract—The linear and non-linear stability of convection of a two-component fluid known as thermohaline convection is considered in a horizontal porous layer heated from below. The analysis is based on the Boussinesq–Darcy equations for 2-dim. convection under the assumption that the amplitudes of convection are small. The linear theory is based on the Fourier analysis and the critical Rayleigh numbers for both marginal and overstable motions are determined. It is found that a vertical solute gradient sets up overstable motions and a physical reason for this is given. The finite amplitude study is based on a truncated representation of Fourier series and the critical Rayleigh number is determined. The effects of Prandtl number, ratio of diffusivities and the permeability parameter on convection are studied. Nusselt number, *Nu*, and its analog Nu^s for solute are calculated and it is found that the effect of Prandtl number is very weak in contrast to the existing viscous flow results.

NOMENCLATURE

- d, depth of the fluid layer;
- g, acceleration due to gravity;
- H, total heat transport;
- $J(f,g)$, $= \partial(f,g)/\partial(x,z)$, Jacobian;
- $\mathbf{k}$, unit vector in the z-direction;
- k, permeability of a porous medium;
- K, the effective thermal conductivity of the porous medium;
- K_s, solute analog of K;
- M, $= (\rho_0 c)_f/(\rho_0 c)^*$;
- Nu, $= Hd/\kappa\Delta T$, Nusselt number;
- Nu^s, $= d/\kappa\Delta s$, solute Nusselt numbers;
- p, $= p_r + ip_i (p_r, p_i > 0)$, frequency;
- P, dynamic pressure;
- P_1^{-1}, $= d^2/k$, porous parameter;
- $\mathbf{q}$, $= (u, v, w)$, mean filter velocity of the fluid;
- R, $= \alpha_T g \Delta T d^3/\nu\kappa$, thermal Rayleigh number;
- R_s, $= \alpha_s g \Delta s d^3/\nu\kappa$, solute Rayleigh number;
- Sc, $= \nu/\kappa_s$, Schmidt number;
- T, temperature;
- T_m, T_0, temperatures of hot and cold walls;
- T_s, T_f, temperatures of the solid and liquid phase;
- x, y, z, space coordinates;
- ∇^2, $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

Greek symbols

- α, the horizontal wave number;
- α_s, solute analog of α_T;
- α_T, the thermal expansion coefficient;
- κ, $= K/(\rho_0 c)_f$, thermal diffusivity;
- κ_s, solute analog of κ;
- ν, the kinematic viscosity of the fluid;
- ρ_0, the mean density;
- (ρc^*), $(\rho c)_f$, $(\rho c)_s$, heat capacities of the porous medium, fluid and solid, $(\rho c)^* = \varepsilon(\rho c)_f + (1-\varepsilon)(\rho c)_s$;
- τ, ratio of diffusivities;
- σ, $= \nu/\kappa$, Prandtl number;
- ε, porosity of the medium;
- ψ, stream function;
- δ^2 $= \pi^2(\alpha^2 + 1)$.

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1. INTRODUCTION

CONVECTIVE hydrothermal reservoirs, the most accessible and well characterized of geothermal resources, are highly permeable where heat and mass transport exist simultaneously. Despite decades of effort, no satisfactory physical or mathematical model exists to study these reservoirs [1]. Any adequate reservoir model would include combined heat and mass transport in a porous medium.

Copious literature [2] is available on the problem of onset of linear and non-linear convection of a single component flow of fluid in a porous medium. A substantial body of literature concerning heat and mass transport in a porous medium also exists. Much of this work relates to the production of crude oil, particularly with regard to *in situ* combustion [3] and hot fluid injection or steam feeding [1]. The most significant phenomenon for oil production, however, is not the same as for steam production. Therefore, the literature related to crude oil production is not directly applicable to the modelling of hydrothermal systems and hence one has to develop a different model by considering natural convection of two-component fluid (known as thermohaline convection) through a porous medium. Here, the buoyancy can arise not only from density differences due to variations in temperature but also from those due to variations in solute concentration. Therefore, the setting up of thermohaline convection in the present analysis will depend upon the destabilizing or stabilizing solute gradient as in the case of viscous flow (i.e. in the absence of a porous medium) [4, 5]. In the case of the destabilizing solute gradient, the configuration becomes unstable because the diffusivity of heat is usually much greater than the diffusivity of the solute and hence a displaced particle of fluid loses any excess heat more rapidly than any excess solute. The resulting buoyancy force may tend to increase the displacement of the particle from its original position and thus cause instability. The same effect may also cause overstability and finite amplitude motions. The case of the stabilizing vertical solute gradient in a layer of fluid can serve to inhibit the onset of convection when the fluid is heated from below for, the Darcy resistance together with the potential energy released by the horizontal temperature gradient are balanced in the absence of inertia (i.e. infinitesimal motion) and local acceleration. The larger the vertical solute gradient the less the potential energy released for a given horizontal temperature gradient to balance with the Darcy resistance. Therefore, the cells will be closer. The same effect may also cause overstability and finite amplitude motions.

The first linear stability analysis of thermohaline convection in a porous medium was performed by Nield [6] and gives the criteria for the existence of steady and oscillatory thermohaline convection. Recently, Rudraiah *et al.* [7] have presented a detailed linear stability analysis of thermohaline convection in a porous layer and have predicted the region of instability via salt-finger and diffusive regimes. The linear theory discussed in [6] and [7] can predict only the conditions for the onset of steady and oscillatory convection and is silent about the heat transport process, because the linear theory cannot predict the amplitude of motion which is the realm of non-linear theory. The non-linear thermohaline convection is comparatively a recent development (see [8]) and has not been given much attention in the case of a porous medium. The chief aim of this paper, therefore, is to consider the non-linear thermohaline convection in a porous medium subject to finite amplitude motion with the object of studying heat transport process. The study is based on the local non-linear stability analysis which is pivoted on the linear theory and in the present paper we concentrate only on the truncated representation of Fourier analysis as explained in [9] and [10]. Specifically the form of velocity, temperature and concentration fields are represented by the linear marginally stable modes plus the first distortion of these modes by the non-linear interaction. No other modes are admitted in the representation. The resulting non-linear equations for the model amplitudes are then solved on the assumption that the motion is steady. Although such an approach involves a drastic simplification of the form of flow fields, especially if the analysis is extended to a value of the Rayleigh number far from the critical value, it does give a physical insight with minimal mathematics. Further, the results of such a study will be useful in the discussion of the fully non-linear problem.

We also note that in the laboratory experiments the effect of concentration gradients is believed to be small since the thermal conduction in a packed bed is usually much more important than the concentration diffusion which only occurs through the void fraction. In geothermal applications, however, the salinity is as high as 2.5×10^5 p.p.m. (see [11]). Therefore, to demonstrate the effects of these on heat transport processes, we consider in this paper a specific example of the aqueous system, heat–sucrose, for which

$$K = 1.4 \times 10^{-3}\,\text{cm}^2\,\text{s}^{-1}, \quad K_s = 0.45 \times 10^{-5}\,\text{cm}^2\,\text{s}^{-1},$$
$$P_r = 10 \quad \text{and} \quad \tau \approx 0.32 = 10^{-1/2}.$$

For such a system R_s is usually $\geqslant 10^7$.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Our discussion is based on an analytical model. The configuration of the convective hydrothermal reservoir to be considered is a horizontal porous layer of uniform thickness d, permeability k, porosity ε and of infinite extent, subject to an adverse temperature gradient ΔT and a stabilizing concentration gradient ΔS. A Cartesian coordinate system has been taken with the origin in the lower boundary and the z-axis vertically upwards. The layer is bounded by two parallel plates at $z = 0$ and $z = d$. The upper plate $z = d$ is at constant temperature $(T_m - \Delta T)$ and concentration $(S_m - \Delta S)$, whereas the lower plate $z = 0$ is at temperature T_m and concentration S_m. We write the total temperature and solute concentration as

$$\begin{aligned} T_{\text{total}} &= T_m - \Delta T\frac{z}{d} + T(x, y, z, t), \\ S_{\text{total}} &= S_m - \Delta S\frac{z}{d} + S(x, y, z, t) \end{aligned} \tag{1}$$

where the first two terms on the right-hand side represent the quiescent state and the last term is due to the convective redistribution.

The boundaries are taken to be dynamically free in the absence of surface tension and are also perfect conductors of heat and solute. The governing equations of motion following [2] are:

the conservation of momentum

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{\nabla p}{\rho_0} + \frac{\rho}{\rho_0}g\hat{k} - \frac{\nu}{k}\mathbf{q}, \tag{2}$$

the conservation of mass

$$\nabla\cdot\mathbf{q} = 0, \tag{3}$$

the conservation of energy

$$(\rho_0 c)^* \frac{\partial T}{\partial t} + (\rho_0 c)_f(\mathbf{q}\cdot\nabla)T = K\nabla^2 T, \tag{4}$$

the conservation of mass flux

$$\frac{\partial S}{\partial t} + (\mathbf{q} \cdot \nabla) S = \kappa_s \nabla^2 S, \tag{5}$$

the equation of state

$$\rho = \rho_0[1 - \alpha_T(T - T_0) + \alpha_s(S - S_0)]. \tag{6}$$

Here K is the effective thermal conductivity of a fluid in the presence of solid matrix which is the sum of the stagnant thermal conductivity due to molecular diffusion and thermal dispersion coefficient due to mechanical dispersion. K_s is the solute analog of K. As far as the thermal equation is concerned, having a given thermal and hydrodynamic state with a motionless fluid phase, i.e. quiescent state for any geometrical point and its associative representative volume, we have to average temperatures T_s and T_f under the assumption that $T_s = T_f = T$. The same explanation holds for the concentration equation also. This is valid when both, solid and fluid phases are well dispersed and if the velocities involved are not too high. This is usually the case in most of the aquifers in geothermal regions. Equations (2)–(5), on making use of the equation of state (6), are made dimensionless using

$$\mathbf{q} = \frac{\kappa}{d}\mathbf{q}', \quad T = T'\Delta T, \quad S = S'\Delta S,$$
$$p' = \frac{pd^2}{\rho_0 \nu \kappa}, \quad (x,z) = d(x',z'), \quad t = d^2/\kappa \cdot t'. \tag{7}$$

Eliminating the pressure by appropriate cross-differentiation of the momentum equation and for simplicity neglecting the primes we get

$$\left(\sigma^{-1}\frac{\partial}{\partial t} + \frac{1}{P_1}\right)\nabla^2\psi = -R\frac{\partial T}{\partial x} + R_s\frac{\partial S}{\partial x} + \sigma^{-1} J(\psi, \nabla^2\psi), \tag{8}$$

$$\left(M^{-1}\frac{\partial}{\partial t} - \nabla^2\right)T + \frac{\partial \psi}{\partial x} = J(\psi, T), \tag{9}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S + \frac{\partial \psi}{\partial x} = J(\psi, S) \tag{10}$$

where the stream function ψ is defined as

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x} \tag{11}$$

and the other quantities have their conventional meanings.

In the case of porous media the thermal Rayleigh number usually involves the permeability k, whereas in the present analysis it is defined independent of k. Further, R_s is defined with κ rather than κ_s in the denominator.

The boundary conditions are

$$\psi = T = S = 0 \quad \text{at} \quad z = 0, 1. \tag{12}$$

3. LINEAR STABILITY ANALYSIS

The chief aim of this paper is to study the finite amplitude thermohaline convection using the local non-linear stability analysis [9] which is pivoted on the linear theory. In this section, we therefore discuss the linear stability analysis briefly considering both marginal and overstable states. A simple physical argument is given for the existence of overstable motions.

The linear stability problem is obtained by setting the Jacobian terms in equations (8)–(10) to zero.

We look for the solutions of the form

$$\left.\begin{aligned}\psi &\sim e^{pt}\sin \pi\alpha x \sin \pi z,\\ T, S &\sim e^{pt}\cos \pi\alpha x \sin \pi z.\end{aligned}\right\} \tag{13}$$

Substituting these into the linearized equations of (8)–(10) and after some simplification we obtain the dispersion relation

$$p^3 + p^2\left[\delta^2(M+\tau) + \frac{\sigma}{P_1}\right] + p\left[\frac{\sigma}{P_1}\delta^2(M+\tau) + \delta^4 M\tau - \frac{\alpha^2\pi^2\sigma}{\delta^2}(MR - R_s)\right] + \alpha^2\pi^2\sigma M(R_s - \tau R) + \frac{\sigma\tau M\delta^4}{P_1} = 0. \tag{14}$$

If p is real, marginal instability occurs when $p = 0$, i.e. when

$$R = R^c = R_s/\tau + \delta^4/\alpha^2\pi^2 P_1. \tag{15}$$

The minimum value of R_c denoted by R^c_m, occurs at

$$\alpha = 1 \tag{16}$$

and its value is

$$R^c_m = R_s/\tau + 4\pi^2/P_1 \tag{17}$$

which is exactly the value given by [4] in the limit of $R_s \to \infty$. Further, we note that when $\tau \to \infty$, or $R_s \to 0$ (i.e. single component fluid), equation (17) leads to

$$R^m_c \to \frac{4\pi^2}{P_1}, \tag{18}$$

the value given by Lapwood [12].

In the case of single component linear stability problem, the horizontal temperature gradient of the perturbed field releases potential energy and the latter is balanced by Darcy resistance offered by the solid particles to the fluid. Thermally, the upward convection of warm fluid is balanced by the diffusion of the excess of temperature. In these simple balances the mean filter velocity and temperature fields are in phase and no restoring force exists; and hence the principle of exchange of stability is valid even in the case of convection through a porous medium. In the case of two-component fluid through a porous medium, however, the principle of exchange of stability is valid only in certain cases as explained below. The concentration field generates a salinity gradient which in the case of linear steady analysis describes a balance between the sum of horizontal temperature gradient and Darcy resistance, see equation (8). This balance is complete because we consider the usual Darcy equation in which the viscous force $\nu\nabla^2\mathbf{q}$ is replaced by $-(\nu/k)\mathbf{q}$. However, the inhibition of convection by concentration field is clearly traceable to the salinity gradient because, a good part of the force which releases potential energy is now balanced by the concentration constraint. The larger the concentration, the larger the salinity gradient. Hence less potential energy is released for a given horizontal temperature gradient. Therefore, time-dependent motions of various types can exist in a two component fluid through a porous medium because the concentration field can act as a restoring mechanism. This time-dependent motion involves a partial balance between the local acceleration and the salinity gradient and hence less salinity gradient is available to offset the horizontal temperature gradient. Convection can therefore be maintained for a smaller imposed temperature difference in the case of overstable motion. For overstable motions $p = p_r + ip_i$. Substituting this into (14) and separating the real and imaginary parts we obtain

$$R \equiv R^o = \frac{R_s(\delta^2\tau + \sigma/P_l)}{M(\delta^2 M + \sigma/P_l)} + \frac{\delta^4(M+\tau)\left[\frac{\sigma}{P_l}\left(\frac{\sigma}{P_l} + \delta^2 M + \delta^2\tau\right) + \delta^4 M\tau\right]}{\pi^2\alpha^2\sigma M(\delta^2 M + \sigma/P_l)}, \quad (19)$$

and

$$p_i^2 = \frac{\sigma}{P_l}\delta^2(M+\tau) + \delta^4 M\tau - \frac{\alpha^2\pi^2\sigma}{\delta^2}(MR - R_s). \quad (20)$$

Since $p_i^2 > 0$, a necessary condition for the existence of overstable motion is

$$R_s \geqslant \frac{\delta^4}{\alpha^2\pi^2 P_l(M-\tau)} \times \left[M\tau\left(\frac{\tau P_l}{\sigma} - 1\right) + (\tau + M)\left(\tau - 2M - \frac{2\sigma}{\delta^2 P_l}\right)\right]. \quad (21)$$

We note that the analytical determination of the minimum wave number is mathematically tedious and is determined numerically. The effect of R_s on the thermal Rayleigh number is studied in Fig. 1 for different values of σ, P_l and τ.

As $R_s \to \infty$, the asymptotic behavior of R^o and p_i^2 are

$$R^o \to \frac{R_s(\delta^2\tau + \sigma/P_l)}{M(\delta^2 M + \sigma/P_l)}, \quad p_i^2 \to \frac{\sigma\alpha^2\pi^2(M-\tau)}{(\delta^2 M + \sigma/P_l)} R_s \quad (22)$$

which shows that $\tau < M$ and

$$\frac{R^o}{R_s} \to \frac{(\delta^2\tau + \sigma/P_l)}{M(\delta^2 M + \sigma/P_l)}. \quad (23)$$

4. FINITE AMPLITUDE STEADY CONVECTION WITH A LIMITED REPRESENTATION

For flows with $R > R_m^c$, the linear stability analysis is not valid and one has to take into account the non-linear effects. In this section, we consider the non-linear stability of convection of a two-component fluid saturated porous layer using a severely truncated representation of Fourier series considering only two terms. Such a study is very useful to understand the physical mechanism with a minimum amount of mathematical analysis and the results can also be used as the starting values while discussing the fully non-linear problem.

The stability problem has a steady solution whose form is given by equation (24) with $p \equiv 0$ for ψ, T and S. The first effect of non-linearity is to distort the temperature and concentration fields through the interaction of ψ and T, and ψ and S respectively. The distortion of the temperature and concentration fields will correspond to a change in the horizontal mean, i.e. a component of the form $\sin 2\pi z$ will be generated.

Therefore, the minimal system which describes finite amplitude convection of a two-component fluid saturated porous layer is given by

$$\psi = A(t)\sin\pi\alpha x \sin\pi z, \quad (24)$$

$$T = B(t)\cos\pi\alpha x \sin\pi z + C(t)\sin 2\pi z, \quad (25)$$

$$S = D(t)\cos\pi\alpha z \sin\pi z + E(t)\sin 2\pi z, \quad (26)$$

where the amplitudes A, B, C, D and E can generally be functions of time t, and are to be determined by the dynamics of the system. Substituting the expressions (24)–(26) into equations (8)–(10) and equating the coefficients of $\sin\pi\alpha x \sin\pi z$, $\cos\pi\alpha x \sin\pi z$ etc., and assuming the amplitudes are steady (i.e. setting $\partial/\partial t = 0$) we obtain the following set of equations as the deterministic set for the steady amplitudes:

$$\frac{\delta^2}{P_l} + \pi\alpha RB - \pi\alpha R_s D = 0, \quad (27)$$

$$\delta^2 B + \pi\alpha A + \pi^2\alpha AC = 0, \quad (28)$$

$$-4\pi^2 C + \tfrac{1}{2}\pi^2\alpha AB = 0, \quad (29)$$

$$\tau\delta^2 D + \pi\alpha A + \pi^2\alpha AE = 0, \quad (30)$$

$$-4\pi^2\tau E + \tfrac{1}{2}\pi^2\alpha AD = 0. \quad (31)$$

These steady solutions are very useful because they predict that a finite amplitude solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of R for which a steady solution is possible lies below the critical values for instability to either a marginal state or an overstable infinitesimal perturbation. Elimination of all amplitudes, except for A, yields after some algebraic simplification

$$A\left\{\frac{\sigma}{P_l}\alpha^4\delta^2\left(\frac{A^2}{8}\right)^2 + \left[\frac{\sigma}{P_l}\frac{\alpha^2}{\pi^2}\delta^2(1+\tau^2)\right.\right.$$

$$+\ \sigma\alpha^4(\tau R_s - R)\Big]\left(\frac{A^2}{8}\right) + \sigma(\alpha^2 + 1)$$

$$\times \left[\frac{\delta^2\tau^2}{P_l} + \alpha^2\tau^2(R_s/\tau - R)\right]\Bigg\} = 0. \quad (32)$$

The solution $A = 0$ corresponds to pure conduction, which we know to be a possible solution though it is unstable when R is sufficiently large. The remaining solutions are given by

$$\frac{A^2}{8} = \frac{-b \pm [b^2 + 4\sigma^2\alpha^6\delta^2\tau^2(R - R^c)]^{1/2}}{2\sigma\alpha^4\delta^2/P_l} \quad (33)$$

where

$$b = \left[(R^c - \tau^{-2}R) + \frac{\delta^4}{\pi^2\alpha^2\tau^2P_l}\right]\sigma\alpha^4\tau^2.$$

Only the solution with the positive sign in front of the radical is admissible, otherwise A^2 is negative, i.e. the amplitude of the stream function is imaginary.

Consider the case where finite solutions exist for $R < R^c$ for a given R_s, σ and τ. The minimum value of R for which solutions exist is denoted by R^f, which makes the radical vanish provided that the first term on the right-hand side of equation (32) be non-negative. The radical vanishes provided that

$$R^f = \left[(\tau R_s)^{1/2} + \frac{\delta^2}{\pi\alpha}\sqrt{\frac{(1-\tau^2)}{P_l}}\right]^2. \quad (34)$$

With this value of R^f, amplitudes are real provided that

$$\frac{1}{P_l} < \frac{\alpha^2 R_s}{\delta^2}\frac{(1-\tau^2)}{\tau^3} \quad \text{and} \quad \tau^2 < 1. \quad (35)$$

We note that R^f_m, the minimum value of R^f at which a steady finite amplitude solution can exist, occurs at $\alpha^2 = 1$ and its value is

$$R^f_m = [(\tau R_s)^{1/2} + 2\pi\sqrt{(1-\tau^2)/P_l}]^2. \quad (36)$$

We note that for a single component fluid $R_s \to 0$ and hence $R^f_m \to 4\pi^2/P_l$ which is the value of R^c_m. In other words in that case both R^c_m and R^f_m coincide and hence subcritical instabilities are not possible and the principle of exchange of stability is valid. To show that subcritical instabilities are possible in the case of a two-component fluid, we compare the values of R^c_m, R^o_m and R^f_m in Fig. 1. The above results are presented in Fig. 1, for $\tau = 10^{-2}$, $\sigma = 10$ and $P_l = 10^{-2}$ to 10^{-6} (approximately the values of salt water), and $\tau = 10^{-1/2}$, $\sigma = 1$ for the same values of P_l.

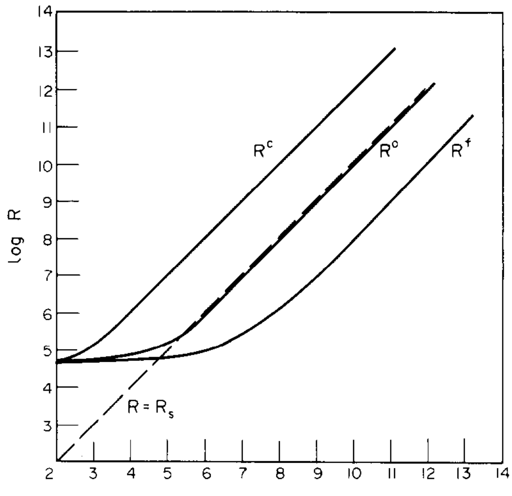

FIG. 1(b). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 10$, $\tau = 10^{-2}$ and $P_l = 0.001$.

Figure 1 predicts the effect of the stabilizing gradient of the solute on the destabilizing adverse temperature gradient. It is evident that, for small values of R_s, the effect of vertical gradient of the solute on the onset of convection is small and the results are close to those of [12], for a single component fluid. As R_s is increased, R^c_m, R^o_m and R^f_m approach asymptotic values for large R_s. Also, the results shown in Fig. 1(e) and 1(f) indicate

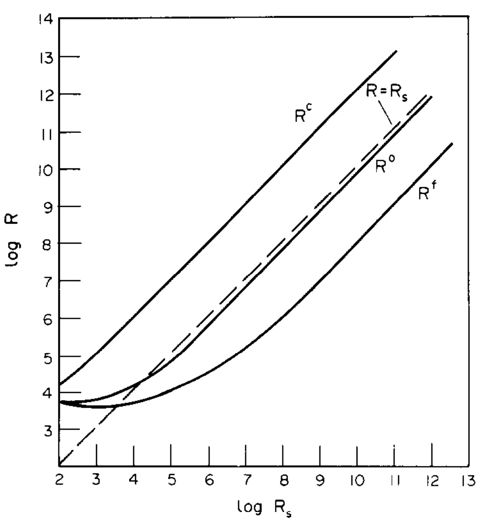

FIG. 1(a). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 10$, $\tau = 10^2$ and $P_l = 0.01$.

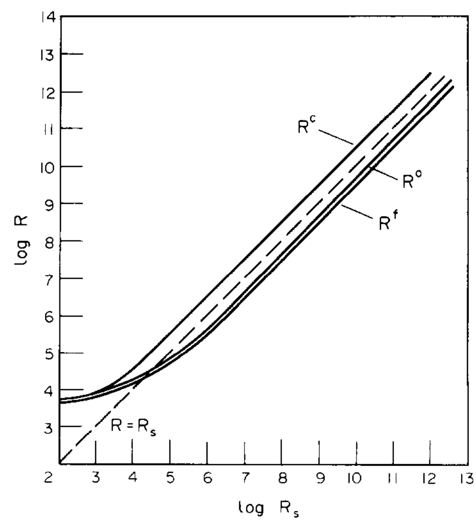

FIG. 1(c). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 1$, $\tau = 10^{-1/2}$ and $P_l = 0.01$.

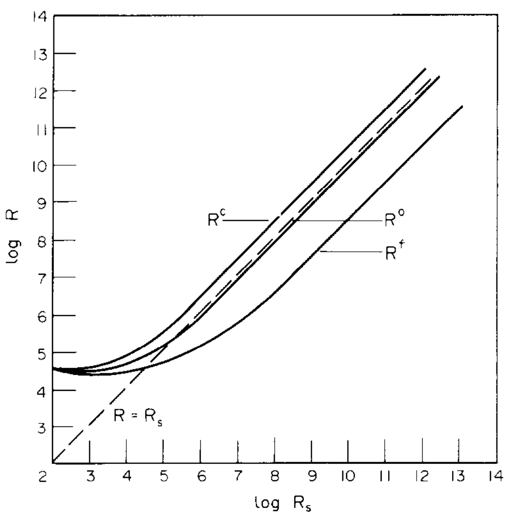

FIG. 1(d). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 1$, $\tau = 10^{-1/2}$ and $P_l = 0.001$.

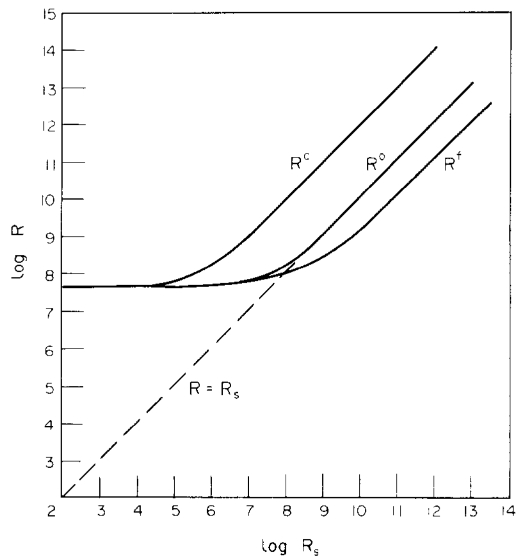

FIG. 1(f). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 10$, $\tau = 10^{-2}$ and $P_l = 10^{-6}$.

that the value of R_s has to be sufficiently large ($\geqslant 10^7$) for the present Darcy–Boussinesq models and the effect of $1/P_l$ is to increase the values of R. For very small values of $P_l (= 10^{-2})$ curves are similar to the viscous case. The line $R = R_s$ corresponds to the curve of neutral buoyancy. This corresponds to the specific example of aqueous system heat–sucrose mentioned at the end of section 1.

5. HEAT TRANSPORT BY CONVECTION

In geothermal regions, the meteoric water percolating down to depth in a permeable formation is heated directly or indirectly by the intruded magma and is then driven buoyantly upward to the top of the aquifer where it can be trapped through drill holes. Therefore, in the study of thermohaline convection the determination of heat transport across the layer plays a very important role. Here, the onset of convection as the Rayleigh number is increased is more rapidly detected by its effect on the heat transfer. In the quiescent state, the heat transfer is usually due to conduction (radiative heat transfer is usually neglected). Hence if H is the rate of heat transfer per unit area

$$H = \kappa \left\langle \frac{\partial}{\partial z} T_{\text{total}} \right\rangle_{z=0} \tag{37}$$

where the angular bracket $\langle \ \rangle$ corresponds to a horizontal average with the definition of T_{total} given by equation (1). Equation (37) can be written in the form

$$H = \kappa \frac{\Delta T}{d} - \kappa \frac{\Delta T}{d} \sum_{n=1}^{N} n\pi b_{\text{on}} \tag{38}$$

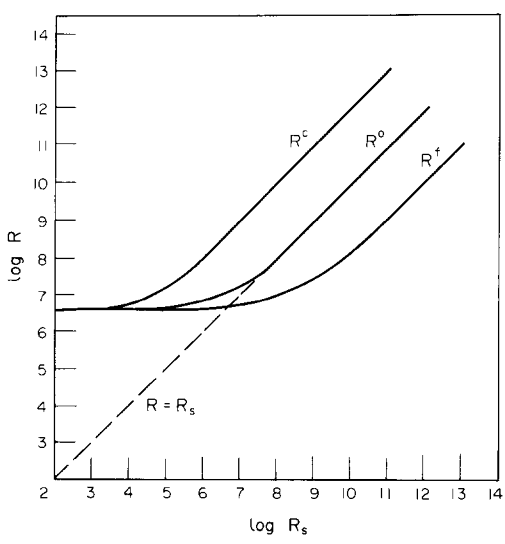

FIG. 1(e). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 10$, $\tau = 10^{-2}$ and $P_l = 10^{-5}$.

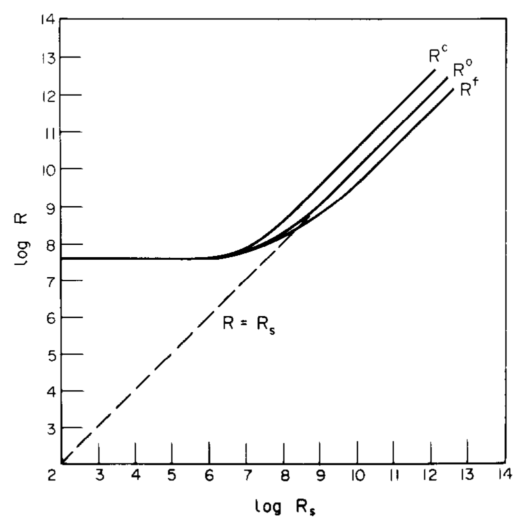

FIG. 1(g). Values of R^c, R^o and R^f are plotted as functions of R_s for the case $\sigma = 1$, $\tau = 10^{-1/2}$ and $P_l = 10^{-6}$.

with restriction $N = 2$. The second term on the right-hand side of (38) represents that the heat which enters at the bottom by conduction is carried on to the top by both conduction and convection and hence the heat transfer increases above that given by conduction alone. This process can be explained physically by the relationship between the driving temperature difference ΔT and the heat transport. In dimensionless variables this is the Rayleigh–Nusselt number curve. The Nusselt number Nu, is the ratio of the heat transported across any layer to the heat which would be transported by conduction alone. Thus from equation (38), the Nusselt number is

$$Nu = \frac{Hd}{\kappa \Delta T} = 1 - \sum_{n=1}^{N} n\pi b_{on} = 1 - 2\pi C \quad (39)$$

with $N = 2$. Similarly, the solute Nusselt number Nu^s, is defined by

$$Nu^s = 1 - \sum_{n=1}^{N} n\pi c_{on} = 1 - 2\pi E \quad (40)$$

where B, C, D and E are given by equations (27)–(31). For Rayleigh numbers below the critical value, the heat transport is purely by conduction for which $A = 0$ and B, C, D and E are all zero. In that case equations (39) and (40) show that Nu and Nu^s have to be unity.

Our object in this study is to determine the effect of the stabilizing gradient of the solute on the destabilizing effect of the adverse temperature gradient. For example, this may correspond to a situation where cold fresh water is injected into the geothermal reservoir containing relatively hot salty water below. The results on stability analysis given in sections 3 and 4 clearly outline the magnitude of the effects of R_s. Thus, when R_s is sufficiently small (much less than $R_c = 4\pi^2/P_l$, the critical Rayleigh number for Lapwood convection with no solute present), the effect of the solute is to modify the results for Lapwood convection by only a small amount. As R_s is increased to this order of R_c, the value of R at which the various types of instability can first occur also increase and as R_s becomes very large, the values of R^c, R^o and R^f approach asymptotically to the values given by equations (17), (22) and (36) respectively. In the pure viscous flow case of Veronis [4] the exact behaviour of the system as a function of R_s depends upon τ as well as σ. Whereas in the present case, the behaviour of the system as a function of R_s depends upon τ and P_l, and the heat transport is independent of σ.

In Table 1, the values of Nu^s and Nu (with Nu^s the upper value in each pair) for $\tau = 10^{-1/2}$, $\tau = 2^{-1/2}$, $R_s = 10^7$ and $P_l = 10^{-5}$ are tabulated and they are independent of σ. As R becomes large, the values of Nu^s and Nu are seen to approach those with no stabilizing gradient ($R_s = 0$) for $\tau = 10^{-1/2}$. The system settles into a steady convective pattern for all $R > 7 \times 10^6$. We see from the table that $R = 7 \times 10^6$ corresponds to conduction and the amount of heat and solute convected by steady modes increase with R. We also note that the effect of τ is to decrease the values of Nu and Nu^s, and when $R_s = 0$, the conduction range is much smaller.

Table 1. Nu^s and Nu vs R and τ

R	$\tau = 10^{-1/2}$	$\tau = 2^{-1/2}$	$R_s = 0$, $\tau = 10^{-1/2}$
4×10^6	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000
5×10^6	1.0000	1.0000	2.2495
	1.0000	1.0000	1.2854
6×10^6	1.0000	1.0000	2.6152
	1.0000	1.0000	1.5913
7×10^6	1.0000	1.0000	2.7493
	1.0000	1.0000	1.8046
8×10^6	2.1126	1.0000	2.8051
	1.2228	1.0000	1.9618
9×10^6	2.5822	1.0000	2.0437
	1.5493	1.0000	2.0824
10^7	2.7267	1.4312	2.8695
	1.7745	1.2416	2.1779
1.519×10^7	2.9023	2.4880	2.9297
	2.3215	2.1847	2.4665
2×10^7	2.9387	2.6845	2.9508
	2.5198	2.4551	2.5972
3×10^7	2.9655	2.8246	2.9696
	2.7014	2.6776	2.7332

$R^o = 15194942.27$.

Since R^f is always sufficiently lower than that of $R^c(=35570634)$, we note that the dependence of Nu and Nu^s on σ is very weak and also instability is first manifested as a finite amplitude steady mode for all σ. The variation of Nu and Nu^s with R is not much. In all cases we note that $Nu^s > Nu$.

Finally we conclude that, although the presence of a stabilizing gradient of solute will serve to inhibit the onset of convection, the strong finite amplitude motions which exist for large Rayleigh numbers tend to mix the solute and distribute it so that the interior layers of the fluid are more neutrally stratified. When this happens, the inhibiting effect of the solute gradient is greatly reduced and the fluid can convect nearly as much heat as it does in the absence of the solute. Therefore, we see that the last column in table which gives the Nusselt number for no solute present coincides with the other columns when R is large.

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CONVECTION D'AMPLITUDE FINIE DANS UNE COUCHE POREUSE SATUREE PAR UN FLUIDE A DEUX COMPOSANTS

Résumé—La stabilité linéaire et non linéaire de la convection par un fluide à deux composants, connue comme convection thermohaline est considérée dans une couche poreuse horizontale chauffée par le bas. L'analyse est basée sur les équations de Boussinesq-Darcy pour la convection bidimensionnelle avec l'hypothèse que les amplitudes de la convection sont petites. La théorie linéaire est basée sur l'analyse de Fourier et les nombres de Rayleigh critiques sont déterminés pour les mouvements marginaux et surstables. On trouve qu'un gradient vertical de soluté accompagne des mouvements surstables et on en donne d'explication physique. L'étude de l'amplitude finie est basée sur une représentation tronquée des séries de Fourier et le nombre de Rayleigh critique est déterminé. Les effets du nombre de Prandtl, du rapport des diffusivités et du paramètre de perméabilité sur la convection sont étudiés. Le nombre de Nusselt Nu et son analogue Nu^s pour le soluté sont calculés et on trouve que l'effet du nombre de Prandtl est très faible en contraste des résultats de l'écoulement visqueux existant.

KONVEKTION MIT ENDLICHER AMPLITUDE IN EINER PORÖSEN SCHICHT, DIE MIT EINER AUS ZWEI KOMPONENTEN BESTEHENDEN FLÜSSIGKEIT GESÄTTIGT IST

Zusammenfassung—Die lineare und nichtlineare Stabilität der Konvektion einer aus zwei Komponenten bestehenden Flüssigkeit, die als thermohaline Konvektion bekannt ist, wird in einer porösen, von unten beheizten Schicht betrachtet. Die Lösung basiert auf den Boussinesq–Darcy-Gleichungen für zweidimensionale Konvektion unter der Annahme, daß die Amplituden der Konvektion klein sind. Das lineare Verfahren basiert auf der Fourier-Analyse, und die kritischen Rayleigh-Zahlen werden sowohl für marginale wie überstabile Bewegungen bestimmt. Es wird festgestellt, daß die Zunahme der Konzentration des gelösten Stoffes mit der Höhe zu überstabilen Bewegungen führt, und es wird eine physikalische Begründung dafür angegeben. Die Untersuchung für endliche Amplituden basiert auf einer abgebrochenen Fourier-Reihen-Entwicklung, wodurch die kritische Rayleigh-Zahl bestimmt wird. Die Einflüsse der Prandtl-Zahl, des Verhältnisses der Diffusionskoeffizienten und der Permeabilitätsparameter auf die Konvektion werden untersucht. Die Nusselt-Zahl Nu und deren Analogon Nu für gelöste Stoffe werden berechnet und gefunden, daß der Einfluß der Prandtl-Zahl im Gegensatz zu bekannten Ergebnissen für zähe Strömung sehr gering ist.

КОНВЕКЦИЯ КОНЕЧНОЙ АМПЛИТУДЫ В ПОРИСТОМ СЛОЕ, НАСЫЩЕННОМ ДВУХКОМПОНЕНТНОЙ ЖИДКОСТЬЮ

Аннотация — Проведено исследование линейной и нелинейной устойчивости конвекции двухкомпонентной жидкости в нагреваемом снизу горизонтальном пористом слое. Анализ основан на уравнениях Буссинеска–Дарси для двухмерной конвекции в предположении малых амплитуд. В результате линейного анализа по методу Фурье определены критические значения числа Релея как для предельных, так и сверхустойчивых движений. Найдено, что вертикальный градиент растворимости приводит к установлению сверхустойчивых движений и дано физическое обоснование этого явления. Исследование конвекции конечной амплитуды проведено с помощью усеченных рядов Фурье. Определено критическое значение числа Релея. Исследовано влияние числа Прандтля, отношения коэффициентов диффузии и параметра проницаемости на конвекцию. Рассчитаны значения числа Нуссельта Nu и его аналога Nu^s для растворенного вещества и найдено, что влияние числа Прандтля несущественно по сравнению со случаем течения вязких жидкостей.